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SEMIGROUP ACTIONS, WEAK ALMOST PERIODICITY, AND INVARIANT MEANS

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Abstract

Let S be a topological semigroup acting on a topological space X. We develop the theory of (weakly) almost periodic functions on X, with respect to S, and form the (weakly) almost periodic compactifications of X and S, with respect to each other. We then consider the notion of an action of S on a Banach space, and on its dual, and after defining S-invariant means for such a space, we give a result concerning the existence of such means, and apply it to prove the existence of a G-invariant mean on the space of weakly almost periodic functions defined on a topological space on which a topological group G acts.

1. Introduction

Let X be a topological space, and let S be a semitopological semigroup acting on X. This means that there exists a mapping $S\times X \longmapsto X$, described by

$$(s,x) \longrightarrow sx \quad (s\epsilon S, x\epsilon X)$$

such that

$$s_1(s_2x)= (s_1s_2)x \qquad (s_1, s_2\epsilon S, x\epsilon S).$$

We take this action to be at least separately continuous. Then X is usually called an S-space, and in particular if S is a topological group G, then X will be called a G-space. Under some conditions the separate continuity of a group or semigroup action or multiplication implies its joint continuity. This is the content of Ellis's theorem and its generalizations which have been described by Ruppert [8]. The references [1], [2], and [8] contain the basic terms, notations, and properties of semigroup actions, specially, those of an action, transitive action, and non-contractive action, which we will use.

In this article, C(X) denotes the space of bounded continuous functions on X, and for $f \epsilon C(X)$, $s \epsilon S$, we define $f_s \epsilon C(X)$ to be the function defined by $f_s(x)= f(sx)$.

The set $O(f,S)=\{f_s: s\epsilon S\}$ is called the orbit of f, with respect to S. We usually abbreviate it to O(f).

In [7] the theory of weakly almost periodic functions on semitopological semigroups has been extended to this more general situation, and relevant compactifications have been formed. It has also been shown that each of the semigroup multiplications and semigroup actions, which coincide if S=X, play a certain role in the properties of function spaces involved.

Since the contents of [7] were not published, a number of its results were independently rediscovered by different authors. The best available sources for most of these are [2] and [8].

In this section we give a brief account of the theory of weakly almost periodic functions on S-spaces.

(1.1) Definition. Let X be an S-space. A function $f\epsilon C(X)$ is called (weakly) almost periodic, with respect to S, if O(f) is (weakly) relatively compact in C (X).

The sets of almost periodic and weakly almost periodic functions, with respect to S, are denoted by AP (X,S)and WAP(X,S) respectively. These notations should not be confused with AP(S,X) and WAP(S,X) used by Goldberg and Irwin [4], which stand for Banach space-valued almost periodic and weakly almost periodic functions on a semigroup; however it should be mentioned that a substantial part of our results can be extended to this situation. If there is no risk of confusion our notations shall be abbreviated to AP(X) and WAP(X).

The following two theorem are the basis of the theory of weakly almost periodic functions. Their proofs are fairly standard, and can be found in [7]. However [2] and [8]contain somewhat different proofs for less general statements.

(1.2) Theorem. AP (X), and WAP (X) are translation invariant, norm closed linear subspaces of C(X), containing constant functions. Furthermore if X is completely regular, they are C^*-subalgebras of C(X).

It should be emphasized that in definition (1.1) and theorem (1.2) the topology of S is, in fact, immaterial.

(1.3) Theorem. Let S be a semitopological semigroup, let X be an S-space, and let f ϵC(X).

(i) If S and X are both compact, then WAP(X) =C(X).

(ii)If S and X are both compact, and the semigroup action is jointly continuous, then AP (X)=C(X).

(iii) If O(f) is relatively norm (weak) compact then the mapping $s \longmapsto f_s$ is norm (weak) continuous.

Notice that in (ii) the joint continuity of semigroup multiplication is not necessary for the conclusion AP(X)=C(X) to hold, and joint continuity of the semigroup action is the essential condition.

(1.4) Corollary. If G is a compact group, acting transitively and separately continuously on the topological space X, then AP (X,G)=C(X).

Proof. We can easily prove that X is a compact space, so, by Ellis's theorem, the group action is jointly continuous. Now apply (ii) of theorem (1.3).

(1.5) Remark. The (weak) almost periodicty of a function f ϵ C(X)does not depend on the topology of S, as the definition of O(f,S) does not involve any topological property of S, and then in taking the (weak) closure of O(f) we do not refer to S at all.

2. Compactifications

We can produce **almost periodic** and **weakly almost periodic compactifications** of either S or X with respect to the other. Naturally when S= X these two different compactifications are isomorphically homeomorphic. The existence of the first compactification follows from the following theorem. The proof can be found in [7], however [2], [3], and [8] contain proofs for less general situations.

(2.1) Theorem. Let B(WAP(X)) be the space of all bounded linear operators from the Banach space WAP(X) (=(WAP (X,S)) into itself. For each sϵS, let T_s be defined on WAP(X) by $T_s f = f_s$. Then the closure of $\{T_s: s\epsilon S\}$ in the weak operator topology of B(WAP(X)), which is denoted by $S^{w,X}$, is a compact topological semigroup.

$S^{w,X}$, will usually be abbreviated to S^w, and called the **weak almost periodic compactification** of S **with respect to** X. Generally speaking the multiplication in S^w is separately continuous.

(2.2) Definition. Since WAP (X) is a C^*-algebra, by the Gelfand-Naimark Theorem we can regard it as the space of continuous functions on a compact space $X_{w,S}$ which is the maximal ideal space of the C^*-algebra WAP(X). X_{wS}, sometimes abbreviated to X_w, is called the **weak almost periodic compactification** of X,**with respect to the semigroups** S.

Parallel to Theorem(2.1)if B (AP(X)) is the space of all bounded operators from AP(X) into itself the closure of $\{T_S: s\epsilon S\}$ in the strong operator topology of B(AP(X)) is a jointly continuous compact semigroup of operators, which will be denoted by $S^{a,X}$ or simply by S^a, and we call it the **almost periodic compactification of** S **with respect to X).**

Also we denote by $X_{a,S}$ the maximal ideal space of the C^*-algebra AP (X). $X_{a,S}$ is called the **almost periodic compactification of X**, with respect to S, and usually denoted by X_a.

Obviously S^a can be identified as a subsemigroup of S^w. Furthermore $s \longmapsto T_s$ defines a homomorphism from S into either of S^a or S^w. If we take X=S, then $S^w = S_w$, and $S^a = S_a$, cf. Burckel [3], p.6.

Since a jointly continuous compact semigroup, containing a dense group is a topological group (see [1], II, 3.6. and [2], III, 9.8), the almost periodic compactification of a group will again be a group.

We can also show that the **multiplication** in S^w is jointly continuous if and only if WAP(X,S)= AP(X,S).

(2.3) Theorem. S^w acts on X_w, and this action is an extension of that of S on X.

Proof. First we define the action of S on X_w. Let X

ϵX_w, then x is a multiplicative linear fuctional on WAP(X). For $s \epsilon S$ and $f \epsilon$ WAP (X) we define (sx)(f) = $x(f_s)$. Since the space of multiplicative linear functionals on a commutative C^*-algebra separates the points, this action is well-defined.

Now let $T \epsilon S^w$, then there exists a net $\{s_\alpha\}_{\alpha \epsilon I}$ such that T_{s_α} tends to T, in the weak operator topology. Take an arbitrary $x \epsilon X_w$ and $f \epsilon$WAP(X)= C(X), then consider the net $\{s_\alpha x\}_{\alpha \epsilon I} \subset X_w$. Since X_w is compact this net has a convergent subnet. We claim that all convergent subnets converge to the same element of X_w. To see this let $\{s_\beta\}$ be a subnet such that $s_\beta x \xrightarrow{\beta} y$. Then $f(s_\beta x) \xrightarrow{\beta} f(y)$. However $f(s_\beta x) = (T_{s_\beta} f)(x)$. Now since $\{T_{s_\beta}\}$ converges to T, we have $f(y) = (Tf)(y)$. Since WAP (X) separates the points of X_w, this will uniquely determine y.

The above construction clearly shows that this action is an extension of the action of S on X.

Similarly we can show that S^a acts on X_a and this action is an extension of that of S on X.

Remark. In view of theorem (2.3), one may ask if we can get an action of S^W on X. However, if S= X, this means that S is an ideal of S^W, which is not always true. For example, take the classical case G= X = $\mathbb{R}$ (the additive group of real numbers), then $\mathbb{R}^w$ is a compact semitopological semigroup, whose identity is the same as the identity of $\mathbb{R}$ (viz. 0), which implies that the group $\mathbb{R}$ can not be an ideal in $\mathbb{R}^w$.

In theorem (1.3) we observed that WAP(X,S) = C(X) if S and X are both compact. The following examples show that the compactness of only one of them is not sufficient to ensure the equality. We first need the following fact which follows from a theorem by A. Grothendieck. See [5], p. 182-183.

Grothendieck's double limit criterion. A function f ϵ C(X) is weakly almost periodic if and only if for each sequence $\{s_n\} \subset S$ and each sequence $\{x_n\} \subset X$ if both $\lim_n \lim_m f(s_m x_n)$ and $\lim_n \lim_m f(s_m x_n)$ exist, then they are equal.

(2.4) Example. There exists a compact space X, and a non-compact group G, such that

$$WAP(X,G) \neq C(X).$$

Proof. Let $\mathbb{Z}$ denote the additive group of integers, and let $\mathbb{Z}$ act on $\mathbb{R}$ by ordinary addition. Define f on $\mathbb{R}$ by

$$f(x) = \begin{cases} kx - k^2 + 1 & \text{for } k - \frac{1}{k} \leqslant x \leqslant k \\ = kx + k^2 + 1 & \text{for } k < x \leqslant k + \frac{1}{k} \\ = 0 & \text{otherwise.} \end{cases} \quad k \geqq 3 \ (k \epsilon \mathbb{Z})$$

Now take as $\{x_n\}$ the sequence $\{\frac{1}{n} : n \epsilon \mathbb{Z}, n > 0\}$, and as $\{s_n\}$ the sequence of positive integers. Then for $m \epsilon \mathbb{Z}$ $n \epsilon \mathbb{N}$, we have $f_m(\frac{1}{n}) = f(m + \frac{1}{n})$. We can easily show that

$$\lim_n \lim_m f(m + \frac{1}{n}) = O \text{ and } \lim_m \lim_n f(m + \frac{1}{n}) = 1,$$

which shows that f is not weakly almost periodic.

Now f extends uniquely to a bounded continuous function f on $\beta \mathbb{R}$, the Stone-Cech compactification of $\mathbb{R}$. On the other hand using the Banach-Stone theorem we can see that the action of $\mathbb{Z}$ on $\mathbb{R}$ extends to an action of $\mathbb{Z}$ on $\beta \mathbb{R}$. We can easily see that the double limit criterion does not hold for f either. i.e.

$$WAP(\beta \mathbb{R}, \mathbb{Z}) = C(\beta \mathbb{R}).$$

Remark. Since $\mathbb{Z}$ is a subsemigroup of $\beta \mathbb{R}$, we can easily see that WAP ($\beta \mathbb{R}$)= WAP ($\beta \mathbb{R}, \beta \mathbb{R}$) $\subset$ WAP ($\beta \mathbb{R}, \mathbb{Z}$). Therefore WAP ($\beta \mathbb{R}$). This shows that separate continuity of the semigroup multiplication is essential for the conclusion of part (i) of theorem (1.3) to hold.

(2.5) Example. There exists a non-compact space X, and a compact group G, such that

$$WAP(X,G) \neq C(X).$$

Proof. Let $\mathbb{R}_1$ be the additive group of real numbers modulo 1, whose addition we denote by $\oplus$, and consider an action of $\mathbb{R}_1$ on $\mathbb{R}_1 \times \mathbb{N}$ defined by

$$t.(u,n) = (t \oplus u, n) \quad (t, u \epsilon \mathbb{R}_1, n \epsilon \mathbb{N}),$$

where $\mathbb{N}$ is the set of positive integers.

Let $f \epsilon C(\mathbb{R}_1 \times \mathbb{N})$ be defined by

$$f(t,n) = \exp(2\pi i n t) \quad (t \epsilon \mathbb{R}_1, n \epsilon \mathbb{N}).$$

If $\{x_n\} \subset \mathbb{R}_1 \times \mathbb{N}$ is defined by $x_n = (0, (2n)!)$, and $\{s_m\} \subset \mathbb{R}_1$ is defined by $s_m = (\frac{1}{2} \sum_{k=m}^{\infty} \frac{1}{k!}) \pmod 1$, then we can prove that

$$\lim_n \lim_m f(s_m x_n) = e^O = 1, \text{ and}$$

$$\lim_m \lim_n f(s_m x_n) = e^i = -1,$$

so, by double limit criterion, f is not weakly almost periodic.

We now face the problem of deciding the conditions under which the equality of WAP (X,S) and C(X) will follow. In the light of Theorem(1.3), and the remark following example (2.4), one may be tempted to conjecture that the equality holds if and only if both X and S are compact, and the semigroup multiplication and possibly the semigroup action are separately continuous. But we can see that a finite S and an arbitrary X will give the equality. Therefore it seems that this, yet open, problem needs a closer inspection, and perhaps some kind of connection between S and X needs to be assumed as well.

3. Invariant means

(3.1) Definition. Let B be a normed linear space, and let S be a semitopological semigroup. We consider an action of S on B, with the following properties:

(i) $s_1(s_2x) = (s_1s_2)x$, $(s_1, s_2 \in S, x \in X)$,

(ii) each $s \in S$ acts linearly on B,

(iii) if S has a unit e then ex = x for all $x \in B$,

(iv) the action is separately continuous when B has its weak topology.

As an example if X is an S-space, then we get an action of S on C(X) defined by

$$(sf)(x) = f(sx) \quad (f \in C(X), s \in S, x \in X),$$

which evidently satisfies the requirements of this definition.

A subset A of B is called **S-invariant** if

$$SA = \{sx: s \in S, x \in A\} \subset A.$$

(3.2) Definition. An element $0 \neq \mu \in B^*$ is called an **S-invariant mean** on B if it has the following properties:

(i) $\|\mu\| = 1$,

(ii) $\mu(sx) = \mu(x)$, $(s \in S, x \in B)$.

If B is a normed vector lattice with a unit, then we demand that $\mu(1) = 1$, or equivalently $\mu \geqq 0$.

It is obvious that if S acts on a normed linear space B, then we can define the action of S on B^*, by

$$(s\phi)(x) = \phi(sx) \quad (s \in S), \phi \in B^*, x \in B).$$

Clearly the action of S on B^* satisfies all the requirements of the definition (3.1), except possibly part (iv), which is not needed in the sequel. However, notice that here the action is separately continuous in the weak*- topology.

(3.3) Definition. With the assumptions of definition (3.1), an element $x \in B$ is called weakly almost periodic, with respect to S, if the set $Sx = \{sx: s \in S\}$ is relatively compact in the weak topology of B. We call x **almost periodic,** with respect to S, if Sx is relatively compact in the norm topology of B.

(3.4) Theorem. Let G be a topological group acting on a Banach space B, and suppose that every $x \in B$ is weakly almost periodic with respect to G. If K is a weak -compact, G-invariant, convex subset of B^*, then K contains a fixed point, under the action of G.

Proof. Consider B^* endowed with the Mackey topology relative to the duality $\langle B, B^* \rangle$. This topology is in fact the topology of uniform convergence on weakly compact subsets of B, and is induced by the set of seminorms.

$$\{P_C: P_C(\phi) = \sup_{x \in C} |\phi(x)| \; ; \phi \in B^*\}$$

where C ranges over the weakly compact convex subsets of B. Since K is weak*- compact, by (4) page 263 of [6], K is compact in the Mackey topology.

Now we prove that G acts non-contractively on B^* (hence K).

Let $\phi \in B^*$ be such that there exists a net $\{g_\alpha\} \subset G$, that $\lim_\alpha g_\alpha \phi = 0$, in the Mackey topology of B^*. This implies that $\lim_\alpha p_c(g_\alpha \phi) = 0$ for each weakly compact convex $C \subset B$. By the Krein-Smulian theorem, for each $x \in B$ the (weakly) closed convex hull of Gx is weakly compact. Thus if we denote the closed convex hull of Gx by $\overline{Gx}$, we have

$$\lim_\alpha P_{\overline{Gx}}(g_\alpha \phi) = \lim_\alpha \sup_{y \in \overline{Gx}} |(g_\alpha \phi(y)| = 0 \quad (x \in B):$$

on the other hand

$$\sup_{y \in \overline{Gx}} |g_\alpha \phi(y)| \geqq \sup_{y \in Gx} |g_\alpha \phi(y)| = \sup_{g \in G} |g_\alpha \phi(gx)|$$

$$= \sup_{g \in G} |\phi(g_\alpha gx)| \geqq |\phi(g_\alpha g_\alpha^{-1} x)| = |\phi(x)|.$$

Therefore

$$\phi(x) = 0 \qquad (x \in X).$$

which implies $\phi \equiv 0$. Now applying Ryll-Nardzewski's theorem, (cf. [1], theorem 2. 16) we obtain the required fixed point.

We now apply this theorem to obtain an extension of a well-known theorem, viz. the existence of an invariant mean on the space of weakly almost periodic functions on a topological group.

(3.5) Theorem. If X is a G-space, then there exists a G-invariant mean on WAP (X,G).

Proof. Let

$$K=\{\mu\in WAP(X,G)': \|\mu\|=\mu(1_X)=1\},$$

where 1_x denotes the constant function on X, whose value at each point of X is 1. We are going to show that this K fulfils the requirements of theorem (3.4).

Obviously K is convex and G-invariant. To show that it is weak*-compact, it is sufficient to show that it is weak*-closed, as it is a subset of the (weak*-compact) unit ball of WAP $(X,G)'$

Suppose that the net $\{\mu_\alpha\}\subset K$ tends to μ in the weak*-topology of WAP $(X,G)^*$. By lower semicontinuity of norms $\|\mu\|=\overline{\lim_\alpha}\|\mu_\alpha\|=1$, while

$$\mu(1_X)=\lim_\alpha \mu(1_X)=1.$$

Therefore $\|\mu\|=\mu(1_X)=1$, i.e. $\mu\in K$.

There are other proofs for the following theorem, but the proof we give, provides a certain connection between the invariant means concerned.

(3.6) Theorem. If X is an S-space where S is a left-amenable semitopological semigroup, in the sense that there exists an invariant mean on C(S), then there exists an S-invariant mean on X.

Proof. Let x be any element of X, and define the mapping π: S ⟶ X by $\pi(s)=sx$. This induces a mapping $\tilde{\pi}$: C(X) ⟶ C(S), which takes the function $f\in C(X)$ to the function $\tilde{\pi}(f)\in C(S)$, defined by

$$\tilde{\pi}(f)(s)=f(sx) \qquad (s\in S).$$

Let $s_1\in S$ be any fixed elements, then for each $s\in S$, we have

$$\pi(f_{s_1})(s)=f_{s_1}(sx)=f(s_1sx);$$

on the other hand

$$(\tilde{\pi}(f))_{s_1}(s)=\tilde{\pi}(f)(s_1s)=(f)(s_1sx),$$

so $(\tilde{\pi}(f))_{s_1}=\tilde{\pi}(f_{s_1})$, i.e. $\tilde{\pi}$ preserves translations by elements of S. We can also see that $\tilde{\pi}(1_x)=1_S$.

Now let $\tilde{\pi}^*$ be the adjoint of $\tilde{\pi}$, and let ϕ be a left invariant mean on C(S). We show that $\tilde{\pi}^*(\phi)$ is an S-invariant mean on C(X).

Let $f\in C(X)$ be a positive function, then $\tilde{\pi}(f)$ is a positive function in C(S). Since ϕ is a mean on S, we have $\phi(\tilde{\pi}(f))\geqq 0$. Therefore

$$(\tilde{\pi}^*(\phi))(f)=\phi(\tilde{\pi}(f))\geqq 0;$$

we also have

$$(\tilde{\pi}^*(\phi))(1_X)=\phi(\tilde{\pi}(1_X))=\phi(1_S)=1.$$

It remains to prove that $\tilde{\pi}(\phi)$ is translation invariant. However

$$(\tilde{\pi}^*(\phi)(f_s))=\phi(\tilde{\pi}(f_s))=\phi((\tilde{\pi}(f))_s)$$
$$=\phi(\tilde{\pi}(f))=(\tilde{\pi}^*(\phi))(f).$$

We can use this theorem to obtain another proof of the theorem (3.5): If G is a topological group then WAP (G) = $C(G^w)$. Since there exists an invariant mean on WAP (G), cf. Burckel [3], Corollary 1.26, this means that G^w is an amenable semigroup. If we suppose that X is a G-space, then, by theorem (2.3), X_w is a G^w-space. Now apply theorem (3.6). This proof has the advantage of producing a G-invariant mean on WAP (X,G), in terms of the invariant mean on WAP (G).

As the reader may have noticed, we have not touched the, yet open, problem of uniqueness of the invariant mean, in the present situation.

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